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HETEROCLINIC TRAVELLING WAVES FOR THE LATTICE SINE-GORDON EQUATION WITH LINEAR PAIR INTERACTION

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ABSTRACT. The existence of travelling heteroclinic waves for the sine-Gordon lattice is proved for a linear interaction of neighbouring atoms. The asymptotic states are chosen such that the action functional is finite. The proof relies on a suitable concentration-compactness argument, which can be shown to hold even though the associated functional has no sub-additive structure.

1. Introduction. We consider the lattice sine-Gordon equation

$$\ddot{q}_k(t) = V'(q_{k+1}(t) - q_k(t)) - V'(q_k(t) - q_{k-1}(t)) - K \sin(q_k(t)), \quad k \in \mathbb{Z}, \quad (1)$$

with a constant $K > 0$. Equation (1) describes the evolution of an infinite chain of atoms with elastic nearest neighbour interaction and an on-site potential, according to Newton's law. The interaction potential $V: \mathbb{R} \rightarrow \mathbb{R}$ takes as argument the discrete strain, which is given by the difference of the positions of the atoms $q_{k+1}(t) - q_k(t)$. In this article, we assume that V is a quadratic function $V(\varepsilon) := \frac{c_0^2}{2}\varepsilon^2$ with $c_0 > 0$ and seek a solution to (1) in the form of a travelling wave by setting $q_k(t) = u(k - ct)$ for $k \in \mathbb{Z}$. Then, a substitution into (1) yields immediately

$$c^2 u''(\tau) = c_0^2(u(\tau + 1) - 2u(\tau) + u(\tau - 1)) - K \sin(u(\tau)). \quad (2)$$

In the setting introduced in Section 2, Equation (2) can be seen to be the Euler-Lagrange equation of the action functional

$$J(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2} (u'(\tau))^2 - \frac{c_0^2}{2} (u(\tau + 1) - u(\tau))^2 + K(1 + \cos(u(\tau))) \right] d\tau. \quad (3)$$

The action functional is the kinetic energy $\int_{\mathbb{R}} \frac{c^2}{2} (u'(\tau))^2 d\tau$ minus the potential energy, consisting of interaction part, $\int_{\mathbb{R}} \frac{c_0^2}{2} (u(\tau + 1) - u(\tau))^2 d\tau$, and on-site part, $\int_{\mathbb{R}} -K(1 + \cos(u(\tau))) d\tau$. This specific choice of the on-site potential is made for simplicity of the presentation; however, all results in this paper can be generalised in a straightforward way to any non-negative, 2π -periodic C^1 -function with zero set $\{(2k + 1)\pi : k \in \mathbb{N}\}$ in place of $(1 + \cos(\cdot))$.

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We are interested in heteroclinic waves (that is, waves that connect two different asymptotic states at $\pm\infty$) for supersonic velocities $c > c_0$. Before stating the precise results, we give a brief overview of some related work.

Bates and Zhang [2] have shown that for a large class of similar models, homoclinic travelling waves exist for supersonic velocities. Their existence result also holds for long-range interaction, but the specialisation to nearest neighbour interaction covers the case

$$c^2 u''(\tau) = c_0^2 (u(\tau + 1) - 2u(\tau) + u(\tau - 1)) + K \sin(u(\tau)). \quad (4)$$

The on-site potential energy can here be taken to be $\int_{\mathbb{R}} [K \cos(u(\tau)) - 1] d\tau$. Bates and Zhang [2] consider homoclinic waves that have their asymptotic states in the maximum of the on-site potential. We study the analogous situation for heteroclinic waves. That is, we consider waves with asymptotic states in two different maxima of the on-site potential. For the choice $-K(1 + \cos(u(\tau)))$ made above for the on-site potential, this leads to the boundary conditions

$$\lim_{\tau \rightarrow -\infty} u(\tau) = -\pi \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} u(\tau) = +\pi. \quad (5)$$

The existence proof will rely on minimisation and a novel type of concentration-compactness. The main difficulties are: Firstly, the action functional, which is to be minimised, is highly nonconvex due to the periodicity of the on-site potential. The second challenge is a lack of compactness due to the infinite domain \mathbb{R} . We show in Section 4 that these difficulties can be overcome with a suitable variant of concentration-compactness [9]. This is not obvious, since the functional (3) is not subadditive. We show that a concentration-compactness result holds nevertheless. This argument relies on the fact that the lattice action functional (3) can be related to the Mortola-Modica functional [10], so that a crucial L^∞ -*a-priori* bound can be inferred. This connection to the Mortola-Modica functional is made explicit in Section 3. Concentration-compactness arguments for lattice models were introduced by Friesecke and Wattis [6] (see also, e.g., [1]).

These two difficulties, namely a highly nonconvex functional and lack of compactness also persist for other boundary conditions, in particular

$$\lim_{\tau \rightarrow -\infty} u(\tau) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} u(\tau) = 2\pi, \quad (6)$$

that is, asymptotic states in the minima of the on-site potential (possibly to be understood in an averaged sense). These boundary conditions correspond to a moving dislocation in the Frenkel-Kontorova model [5]. The existence of periodic solutions and sliding solutions for the two-dimensional generalisation of the Frenkel-Kontorova model can be shown with topological and variational methods [4]. A survey over some related results can be found in the book by Pankov [11]. For the one-dimensional Frenkel-Kontorova model, there are existence results for heteroclinic waves with asymptotic states (6) for the special case of a piecewise quadratic on-site potential in the physics literature [8]. There, it is assumed that the solution satisfies the sign condition of the kind

$$u(\tau) < \pi \text{ for } \tau < 0 \quad \text{and} \quad u(\tau) > \pi \text{ for } \tau > 0. \quad (7)$$

Under this assumption, the analogue of the Euler-Lagrange equation (2) for piecewise on-site potential simplifies to an equation with a nonlinearity that depends only on τ , rather than $u(\tau)$. This simplified system is then solved by Fourier methods, where the solution is represented as a sum of Fourier components. The difficulty

is to show that the solution satisfies the sign condition (7). Kresse and Truskivsky [8] observe that this condition probably does not hold for a specific interval of subsonic velocities. A rigorous proof that the sign condition holds in some regime seems, at the time of writing, only to be available for the Fermi-Pasta-Ulam chain with piecewise quadratic pair interaction [12]. The extension of this result to more general potentials is an open problem.

2. Main result. We set $X := \{u \in H_{\text{loc}}^1(\mathbb{R}) : u' \in L^2(\mathbb{R})\}$ and remark that X is a Hilbert space when equipped with the inner product

$$\langle u, v \rangle_X := u(0) v(0) + \int_{\mathbb{R}} u'(\tau) v'(\tau) \, d\tau.$$

Further, let us define

$$\mathcal{M}_{-\pi, \pi} := \{u \in X : u(-\infty) = -\pi, u(\infty) = \pi\}. \quad (8)$$

We are now in a position to formalise the connection of Equation (2) and the action functional $J : X \rightarrow \mathbb{R} \cup \{\infty\}$ given in (3).

Let $v_0 : \mathbb{R} \rightarrow [-\pi, \pi]$ be a monotone function in $C^\infty(\mathbb{R})$ such that $v_0(\tau) = -\pi$ for $\tau < -1$ and $v_0(\tau) = \pi$ for $\tau > 1$ and define $\Psi : H^1(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\Psi(v) := J(v_0 + v).$$

It is not hard to see that $\Psi(v) < \infty$ for all $v \in H^1(\mathbb{R})$, and that, conversely, a minimiser u of J on $\mathcal{M}_{-\pi, \pi}$ can be written as $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$ (for details see [7]). Furthermore, Ψ is continuously differentiable on $H^1(\mathbb{R})$.

Lemma 2.1 (Euler-Lagrange equation and regularity). *Suppose $v \in H^1(\mathbb{R})$ is a critical point of Ψ ; set $u := v_0 + v \in \mathcal{M}_{-\pi, \pi} \subset X$. Then $u \in C^2(\mathbb{R})$ and u is a solution of (2) with boundary conditions (5).*

Proof. Every critical point $v \in H^1(\mathbb{R})$ of Ψ satisfies by definition $\langle \Psi'(v), h \rangle = 0$ for all $h \in H^1(\mathbb{R})$, that is,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} [c^2 u'(\tau) h'(\tau) - c_0^2 (u(\tau+1) - u(\tau))(h(\tau+1) - h(\tau)) - K \sin(u(\tau)) h(\tau)] \, d\tau \\ &= \int_{\mathbb{R}} [c^2 u'(\tau) h'(\tau) + c_0^2 [u(\tau-1) - 2u(\tau) + u(\tau+1)] h(\tau) - K \sin(u(\tau)) h(\tau)] \, d\tau. \end{aligned}$$

This means that u is a weak solution of (2). Applying a classical bootstrap argument, we find that $u \in C^2(\mathbb{R})$ is a strong solution of (2). \square

Lemma 2.1 shows in particular that a minimiser of the variational problem

$$\text{minimise } J, \text{ as defined in (3), on } \mathcal{M}_{-\pi, \pi} \subset X \quad (9)$$

is a solution to (2) with boundary conditions (5).

We now formulate the existence result for (2), for sufficiently large supersonic wave speed and heteroclinic boundary conditions.

Theorem 2.2. *Let $c^2 > \frac{9}{8} c_0^2$. Then there exists a minimiser u_0 of J on $\mathcal{M}_{-\pi, \pi} \subset X$, that is, the variational problem (9) possesses a solution. This minimiser u_0 is a C^2 -function which satisfies (2) and the asymptotic boundary condition (5).*

The proof of this Theorem will follow easily from Lemma 2.1 and the statements in Sections 3 and 4; it is given in Section 5.

3. A-priori bound. For more a compact notation, we introduce on X a difference operator A as $Au(z) := u(z+1) - u(z)$. Observe that for fixed $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$,

$$\int_{T_1}^{T_2} [Au(\tau)]^2 d\tau \leq \int_0^1 \int_{T_1+s}^{T_2+s} [u'(\tau)]^2 d\tau ds; \quad (10)$$

this follows with Jensen's inequality and Fubini's theorem,

$$\begin{aligned} \int_{T_1}^{T_2} [Au(\tau)]^2 d\tau &= \int_{T_1}^{T_2} \left[\int_{\tau}^{\tau+1} u'(s) ds \right]^2 d\tau = \int_{T_1}^{T_2} \left[\int_0^1 u'(t+\tau) dt \right]^2 d\tau \\ &\leq \int_{T_1}^{T_2} \int_0^1 [u'(t+\tau)]^2 dt d\tau = \int_0^1 \int_{T_1+t}^{T_2+t} [u'(\zeta)]^2 d\zeta dt. \end{aligned}$$

By the same argument, $\int_{\mathbb{R}} [Au(\tau)]^2 d\tau \leq \int_{\mathbb{R}} [u'(\tau)]^2 d\tau$ (see also [13]). This implies

$$\begin{aligned} \int_{\mathbb{R}} \left[\frac{c^2 - c_0^2}{2} (u'(\tau))^2 + K(1 + \cos(u(\tau))) \right] d\tau \\ \leq J(u) \leq \int_{\mathbb{R}} \left[\frac{c^2}{2} (u'(\tau))^2 + K(1 + \cos(u(\tau))) \right] d\tau. \end{aligned}$$

for all $u \in X$. Modica and Mortola [10] have studied a very similar functional to those in this inequality. We quote a relevant result on the minimal values of such functionals from [3, Section 6.2].

Lemma 3.1. *For $\gamma > 0$, let $I_\gamma(u) := \int_{\mathbb{R}} [\gamma (u'(\tau))^2 + K(1 + \cos(u(\tau)))] d\tau$. Then the minimum of I_γ on $\mathcal{M}_{-\pi, \pi}$ is attained and*

$$\min_{u \in \mathcal{M}_{-\pi, \pi}} I_\gamma(u) = \vartheta := 2\sqrt{\gamma K} \int_{-\pi}^{\pi} \sqrt{1 + \cos(\xi)} d\xi. \quad (11)$$

Moreover, with the same ϑ ,

$$\inf_{T>0} \inf \left\{ \int_{-T}^T [\gamma (u')^2 + K(1 + \cos(u))] d\tau : \begin{array}{l} u \in H^1(-T, T), \\ u(-T) = -\pi, u(T) = \pi \end{array} \right\} = \vartheta. \quad (12)$$

As an immediate consequence we get, by evaluating the integral in (11),

$$8\sqrt{(c^2 - c_0^2)K} \leq \inf_{u \in \mathcal{M}_{-\pi, \pi}} J(u) \leq 8c\sqrt{K}. \quad (13)$$

This inequality and (12) will serve as basis for the L^∞ -a-priori bound in the next lemma.

Lemma 3.2. *Let $c^2 > c_0^2$. A global minimiser u_0 of J on $\mathcal{M}_{-\pi, \pi}$ satisfies*

$$\|u_0\|_{L^\infty(\mathbb{R})} < (2k+3)\pi,$$

$$\text{where } k := \max \left\{ \kappa \in \mathbb{N}_0 : (2\kappa+1) \leq \sqrt{\frac{c^2}{c^2 - c_0^2}} \right\}.$$

Proof. The proof relies on the fact that (12) and (13) provide, loosely speaking, an estimate for the “cost” for u_0 to traverse a height of 2π from one minimum of $\cos(\cdot)$ to the next. More precisely, we have

$$J(u_0) \geq I_{\frac{1}{2}(c^2 - c_0^2)}(u_0) := \int_{\mathbb{R}} \left[\frac{c^2 - c_0^2}{2} [u_0'(\tau)]^2 + K [1 + \cos(u_0(\tau))] \right] d\tau. \quad (14)$$

Hence, (13) and $1 + \cos(u) \geq 0$ show

$$\frac{c^2 - c_0^2}{2} \|u'_0\|_{L^2(\mathbb{R})}^2 \leq J(u_0) \leq 8c\sqrt{K}. \quad (15)$$

Let $T_1, T_2 \in \mathbb{R} \cup \{\pm\infty\}$ with $T_1 < T_2$ be such that $u_0(T_1) = \nu\pi$ and $u_0(T_2) = (\nu + 2)\pi$ for some odd integer ν . Then the contribution of the interval $[T_1, T_2]$ to the value $I_{(c^2 - c_0^2)}(u_0)$ is, from (12),

$$\begin{aligned} & \int_{T_1}^{T_2} \left[\frac{c^2 - c_0^2}{2} [u'_0(\tau)]^2 + K [1 + \cos(u_0(\tau))] \right] d\tau \\ & \geq \inf_{\substack{u \in H_{\text{loc}}^1(T_1, T_2) \cap C^0[T_1, T_2], \\ u(T_1) = \nu\pi, \quad u(T_2) = (\nu+2)\pi}} \int_{T_1}^{T_2} \left[\frac{c^2 - c_0^2}{2} [u'(\tau)]^2 + K [1 + \cos(u(\tau))] \right] d\tau \\ & \geq 8\sqrt{(c^2 - c_0^2)K}. \end{aligned} \quad (16)$$

The boundary conditions (5) imply that the height 2π needs to be covered; any further increase in height of 2π has to be compensated by a decrease in height of 2π and vice versa. Hence there is an odd number of such increases or decreases. We write this odd number as $2\kappa + 1$ with $\kappa \in \mathbb{N}$, so that $(\kappa + 1) \cdot 2\pi \leq |\{u_0(z) : z \in \mathbb{R}\}| < (\kappa + 2) \cdot 2\pi$; thus κ can be understood as a lower bound on the number of times that u_0 grows by full 2π in excess to the one time required by $u_0 \in \mathcal{M}_{-\pi, \pi}$. Then (14) and (16) show that

$$J(u_0) \geq I_{\frac{1}{2}(c^2 - c_0^2)}(u_0) \geq 8(2\kappa + 1)\sqrt{(c^2 - c_0^2)K}.$$

On the other hand, $J(u_0)$ is bounded by the Modica-Mortola bound (13). Therefore, using (15),

$$8(2\kappa + 1)\sqrt{(c^2 - c_0^2)K} \leq J(u_0) \leq 8c\sqrt{K} \quad (17)$$

so that

$$\kappa \leq k := \max \left\{ \kappa \in \mathbb{N}_0 : (2\kappa + 1) \leq \sqrt{\frac{c^2}{c^2 - c_0^2}} \right\}.$$

Hence $(k + 1) \cdot 2\pi \leq |\{u_0(z) : z \in \mathbb{R}\}| < (k + 2) \cdot 2\pi$. Due to (5), $(-\pi, \pi) \subseteq \{u_0(z) : z \in \mathbb{R}\}$, so

$$\sup_{\tau \in \mathbb{R}} |u_0(\tau)| < (k + 2) \cdot 2\pi - \pi = (2k + 3)\pi;$$

note in particular that the inequality is strict. \square

4. Concentration-compactness. The next step is to prove a variant of the concentration-compactness lemma of P.-L. Lions [9, Lemma I.1] that is adapted to our situation.

The setting in this classical paper [9] (see also [6]) is as follows. The general problem is to minimise a functional $E: U \rightarrow \mathbb{R}$ on a Banach space U subject to a constraint $L(u) = \lambda > 0$. It is shown that, for fixed λ , that any minimising sequence is, up to a subsequence, either relatively compact, or vanishes, or splits into two or more parts which drift away arbitrarily distant from each other. Vanishing can usually be excluded quite easily. Setting

$$I_\lambda := \inf \{E(u) : u \in U, L(u) = \lambda\},$$

splitting cannot occur, heuristically speaking, if and only if

$$I_\lambda < I_\alpha + I_{\lambda-\alpha} \text{ for all } \alpha \in (0, \lambda).$$

In comparison to the classical setting, a major difference in the present paper is that the constraint $u(\pm\infty) = \pm\pi$ cannot be varied continuously. Hence it is impossible to consider the minimum value of the functional on level sets of the constraint as a function of a continuous parameter in the constraint. As a consequence of this, no meaningful analog to the above subadditivity inequality can be formulated. Instead, we will exclude splitting by means of the *a priori* bound from Lemma 3.2.

The most important difference in contrast to other variants of the concentration-compactness lemma is therefore in the alternative of splitting. The value of the functional J is split up between sequences $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}$ (whose sum is essentially the original sequence $(u_n)_{n \in \mathbb{N}}$)—not the value of the constraint, as usual. On the other hand, the present lemma holds not just for minimising sequences $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$, but for all sequences for which the values of the functional converge.

The following proof will be formulated using symmetrised differences $u(\tau + \frac{1}{2}) - u(\tau - \frac{1}{2})$, rather than $u(\tau + 1) - u(\tau)$, in order to exploit the symmetry of the integration domains. It is clear that J , and hence the minimisation problem, remains unchanged because $\int_{\mathbb{R}} [Au(\tau)]^2 d\tau = \int_{\mathbb{R}} [u(\tau + \frac{1}{2}) - u(\tau - \frac{1}{2})]^2 d\tau$.

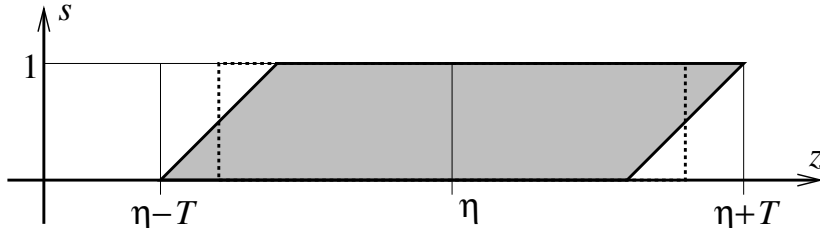


FIGURE 1. Domain of integration of $\int_0^1 \int_{\eta-T+s}^{\eta+T-1+s} [u'(\tau)]^2 d\tau ds$, which is up to a multiplicative factor the first term in the definition of $J_T(\cdot; \eta)$ in (18).

We introduce a truncated version of J . For parameters $T > 1$ and $\eta \in \mathbb{R}$, set

$$\begin{aligned} J_T(u; \eta) := & \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta-T+\frac{1}{2}+s}^{\eta+T-\frac{1}{2}+s} \frac{c^2}{2} [u'(\tau)]^2 d\tau ds \\ & - \int_{\eta-T+\frac{1}{2}}^{\eta+T-\frac{1}{2}} \frac{c_0^2}{2} [u(\tau + \frac{1}{2}) - u(\tau - \frac{1}{2})]^2 d\tau + \int_{\eta-T+\frac{1}{2}}^{\eta+T-\frac{1}{2}} K [1 + \cos(u(\tau))] d\tau \quad (18) \end{aligned}$$

We point out that all integrals are taken over symmetric intervals around η which simplifies some estimates later in this proof. For use in Section 5, we mention that the second summand equals $\int_{\eta-T}^{\eta+T-1} [Au(\tau)]^2 d\tau$.

The domain of integration for the first term in the definition of J_T can be thought of as the shaded parallelogram shown in Figure 1, with the integrand being constant

on vertical lines. This choice is motivated by the second term because

$$Au(\tau) = \int_{\tau}^{\tau+1} u'(s) \, ds$$

shows that, roughly speaking, the second term could be interpreted as an integration over the same domain. This idea has already been suggested by (10).

Lemma 4.1 (Concentration-compactness). *Let $c^2 > c_0^2$ and $\theta \geq \inf J(u)|_{\mathcal{M}_{-\pi,\pi}}$, $\theta < \infty$. Then every sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$ with the property*

$$\lim_{n \rightarrow \infty} J(u_n) = \theta \quad (19)$$

possesses a subsequence, not relabelled and still denoted by $(u_n)_{n \in \mathbb{N}}$, which satisfies one of the following three alternatives:

- (i) Tightness: *There is a sequence $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that, for every $\varepsilon > 0$, there exists $T_0 > 0$ such that for all $T > T_0$*

$$J(u_n) - J_T(u_n; \eta_n) < \varepsilon \text{ for every } n \in \mathbb{N}.$$

- (ii) Vanishing: *For all $T > 0$,*

$$\lim_{n \rightarrow \infty} \sup_{\eta \in \mathbb{R}} J_T(u_n; \eta) = 0. \quad (20)$$

- (iii) Splitting: *There exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, there are $f_n, g_n \in X$ such that*

$$\begin{aligned} |u_n - (f_n + g_n - \pi)| &\leq \varepsilon \\ |J(u_n) - (J(f_n) + J(g_n))| &\leq \varepsilon, \quad \lim_{n \rightarrow \infty} \text{dist}(\text{supp}(f'_n), \text{supp}(g'_n)) = \infty, \\ \lim_{n \rightarrow \infty} J(f_n) &= \alpha, \quad \lim_{n \rightarrow \infty} J(g_n) = \beta, \end{aligned}$$

for some $0 < \alpha, \beta < \theta$. (π is needed in the first inequality to ensure $J(f_n) < \infty$ and $J(g_n) < \infty$.)

The condition (19) is in particular satisfied if $(u_n)_{n \in \mathbb{N}}$ is a minimising sequence for J . We actually need Lemma 4.1 only for that case, in which $\theta = \inf J(u)|_{\mathcal{M}_{-\pi,\pi}}$.

Proof. The proof is given in four steps. First we introduce a concentration functional, discuss its properties (Step 1). The rest is concerned with the proof that the only alternative to cases (i) and (ii) is case (iii). Step 2 identifies the intervals which will become the support of f_n and g_n , respectively. Further estimates show the statements about the sequences f_n (Step 3) and g_n (Step 4).

Step 1. As in Lions' proof [9], a concentration function is introduced. Namely, given a sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$ with (19) and a parameter $\eta \in \mathbb{R}$, define a sequence of functions $P_n(\cdot; \eta): (0, \infty) \rightarrow \mathbb{R}$,

$$P_n(T; \eta) := J_{T+\frac{1}{2}}(u_n; \eta) \quad (\text{compare Figure 1}) \quad (21)$$

the shift by $\frac{1}{2}$ from the definition of J_T saves some summands $\frac{1}{2}$ in subsequent estimates of this proof while the form of J_T is more useful in Section 5 below.

Note that, for every fixed $n \in \mathbb{N}$ and $\eta \in \mathbb{R}$, P_n is nondecreasing in T . Namely, for all $\varepsilon > 0$ and all $\eta \in \mathbb{R}$, (10) and $c > c_0$ show that the increment of the second

integral in (21) is bounded by the increment of the first one,

$$\begin{aligned} \frac{c_0^2}{2} \int_{\eta+T}^{\eta+T+\varepsilon} \left[u_n \left(\tau + \frac{1}{2} \right) - u_n \left(\tau - \frac{1}{2} \right) \right]^2 d\tau &= \frac{c_0^2}{2} \|Au_n\|_{(\eta+T-\frac{1}{2}, \eta+T-\frac{1}{2}+\varepsilon)}^2 \\ &\leq \frac{c^2}{2} \int_0^1 \int_{\eta+T-\frac{1}{2}+s}^{\eta+T-\frac{1}{2}+\varepsilon+s} [u'_n(\tau)]^2 d\tau ds = \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta+T+s}^{\eta+T+\varepsilon+s} [u'_n(\tau)]^2 d\tau ds, \end{aligned} \quad (22)$$

and the very same estimate holds on $(\eta - T - \varepsilon, \eta - T)$. This implies $P_n(T + \varepsilon; \eta) \geq P_n(T; \eta)$ for all $T, \varepsilon > 0$ because $1 + \cos(u_n)$ is always non-negative.

Now we can define for each $n \in \mathbb{N}$ the concentration function

$$Q_n(T) := \sup_{\eta \in \mathbb{R}} P_n(T; \eta). \quad (23)$$

As supremum of monotone and nondecreasing functions, Q_n enjoys the same properties. It is clear that Q_n is bounded on $(0, \infty)$ because, for each $n \in \mathbb{N}$,

$$\lim_{T \rightarrow \infty} Q_n(T) = J(u_n).$$

By assumption (19), $(J(u_n))_{n \in \mathbb{N}}$ converges to θ and is therefore *a fortiori* bounded in \mathbb{R} ; thus the sequence $(Q_n)_{n \in \mathbb{N}}$ is bounded from above in $L^\infty(0, \infty)$. Hence, by Helly's selection theorem (see, e.g., [14, Section 17.4]), a subsequence, not relabelled, converges pointwise almost everywhere to a monotone nondecreasing function $Q: (0, \infty) \rightarrow \mathbb{R}$ and

$$l := \lim_{T \rightarrow \infty} Q(T) \in [0, \theta]. \quad (24)$$

Obviously, alternative (i) in the statement occurs for $l = \theta$, and alternative (ii), vanishing, occurs when $l = 0$. What remains is to show that $0 < l < \theta$ corresponds to alternative (iii), splitting.

Step 2. Let $\varepsilon > 0$. By definition of l in (24), there exists $T_0 \in \mathbb{R}$ such that $Q(T_0) \geq l - \frac{1}{3}\varepsilon$. Since $Q_n(T) \rightarrow Q(T)$ as $n \rightarrow \infty$ for almost every T , we may assume, possibly after increasing T_0 , that $Q_n(T_0) \rightarrow Q(T_0)$. Thus, $Q_n(T_0) \geq l - \frac{2}{3}\varepsilon$, if we consider only large enough n . The definition (23) of Q_n implies that we can find $\eta_n \in \mathbb{R}$ such that for all large enough n

$$P_n(T_0; \eta_n) \geq l - \varepsilon.$$

We can also find a sequence $(T_n)_{n \in \mathbb{N}}$ with $T_n \rightarrow \infty$ as $n \rightarrow \infty$ (and in particular $T_n \gg T_0$ for all $n \in \mathbb{N}$) such that $Q_n(T_n) \leq l + \varepsilon$; this follows from the facts that $Q_n(T) \rightarrow Q(T)$ as $n \rightarrow \infty$ for almost every T , and that $Q(T) \rightarrow l$ as $T \rightarrow \infty$, see (24). Since Q_n has been defined as supremum over P_n in (23), the sequence $(T_n)_{n \in \mathbb{N}}$ satisfies $P_n(T_n; \eta_n) \leq l + \varepsilon$. As P_n is monotone and nondecreasing in T for each $n \in \mathbb{N}$,

$$|P_n(T; \eta_n) - l| \leq \varepsilon \text{ for all } T \in [T_0, T_n]. \quad (25)$$

Now we are going to analyse the behaviour of $u_n(\tau)$ for $|\tau - \eta_n| \in [T_0, T_n]$; the goal is to show that there exist $k_n^\pm \in \mathbb{Z}$ such that

$$\begin{aligned} |u_n(\tau) - (2k_n^+ + 1)\pi| &\leq \delta(\varepsilon) \text{ for } \tau \in [\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}] \text{ and} \\ |u_n(\tau) - (2k_n^- + 1)\pi| &\leq \delta(\varepsilon) \text{ for } \tau \in [\eta_n - T_n + \frac{1}{2}, \eta_n - T_0 - \frac{1}{2}] \end{aligned} \quad (26)$$

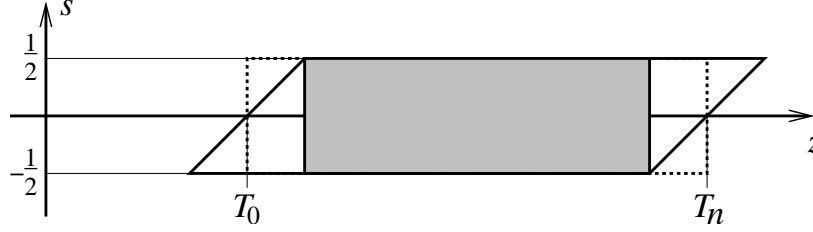


FIGURE 2. Domains of integration of the first term (parallelogram) and the second term (dashed rectangle) in (27) and of the integral in the following line (shaded rectangle). To prevent a complicated labelling of axes, the situation has been sketched for $\eta_n = 0$.

Starting with (25) and considering first only the interval $[\eta_n + T_0, \eta_n + T_n]$, we find, first arguing as for the derivation of the inequality in (22) and then shrinking the domain of integration in the second step (compare Figure 2),

$$\begin{aligned}
2\varepsilon &\geq P_n(T_n; \eta_n) - P_n(T_0; \eta_n) \\
&\geq \frac{c^2 - c_0^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta_n + T_0 + s}^{\eta_n + T_n + s} [u'_n(\tau)]^2 d\tau ds + K \int_{\eta_n + T_0}^{\eta_n + T_n} [1 + \cos u_n(\tau)] d\tau \\
&\geq \int_{\eta_n + T_0 + \frac{1}{2}}^{\eta_n + T_n - \frac{1}{2}} \left[\frac{c^2 - c_0^2}{2} [u'_n(\tau)]^2 + K [1 + \cos(u_n(\tau))] \right] d\tau \tag{27}
\end{aligned}$$

and, for any $T', T'' \in \mathbb{R}$ with $T_0 + \frac{1}{2} \leq T' < T'' \leq T_n - \frac{1}{2}$, by shrinking the domain of integration again,

$$\geq \int_{\eta_n + T'}^{\eta_n + T''} \left[\frac{c^2 - c_0^2}{2} [u'_n(\tau)]^2 + K [1 + \cos(u_n(\tau))] \right] d\tau,$$

and using the trivial estimate $x^2 + y^2 \geq 2xy$ and the change of variables $\xi = u_n(\tau)$,

$$\begin{aligned}
&\geq 2 \int_{\eta_n + T'}^{\eta_n + T''} \left[\sqrt{\frac{c^2 - c_0^2}{2}} |u'_n(\tau)| \cdot \sqrt{K(1 + \cos(u_n(\tau)))} \right] d\tau \\
&= \sqrt{2K(c^2 - c_0^2)} \left| \int_{u_n(\eta_n + T')}^{u_n(\eta_n + T'')} \sqrt{1 + \cos(\xi)} d\xi \right|.
\end{aligned}$$

This shows that $|u_n(\eta_n + T') - u_n(\eta_n + T'')| \leq \delta_1$, where $\delta_1 = \delta_1(\varepsilon)$ is given by the relation

$$\int_{\pi - \frac{\delta_1}{2}}^{\pi + \frac{\delta_1}{2}} \sqrt{1 + \cos(x)} dx = \frac{2\varepsilon}{\sqrt{2K(c^2 - c_0^2)}}.$$

To see that this interval $\{u_n(\tau) : \tau \in [\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}]\}$ of length $\leq \delta_1$ is near an element of $\{(2k+1)\pi : k \in \mathbb{Z}\}$, observe that (27) implies also

$$2\varepsilon \geq K(T_n - T_0 - 1) \cdot \min_{\tau \in [\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}]} [1 + \cos(u_n(\tau))].$$

Suppose this minimum is attained at $\tau_{0,n}$. Then $\frac{2\varepsilon}{K(T_n - T_0 - 1)} \geq 1 + \cos(u_n(\tau_{0,n}))$. As we may assume $\varepsilon \ll 1$ and $T_n \gg T_0$, there exists a k_n^+ such that

$$|u_n(\tau_{0,n}) - (2k_n^+ + 1)\pi| \leq \pi - \arccos\left(\frac{2\varepsilon}{K(T_n - T_0 - 1)} - 1\right) =: \delta_2$$

Hence, with $\delta = \delta(\varepsilon) = \delta_1 + \delta_2$,

$$|u_n(\tau) - (2k_n^+ + 1)\pi| \leq \delta \quad \text{for all } \tau \in [\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}],$$

and $\delta \rightarrow 0$ for $\varepsilon \rightarrow 0$. To establish (26), it suffices to observe that the same argument yields k_n^- with the corresponding property

$$|u_n(\tau) - (2k_n^- + 1)\pi| \leq \delta \quad \text{for all } \tau \in [\eta_n - T_n + \frac{1}{2}, \eta_n - T_0 - \frac{1}{2}].$$

Step 3. Define

$$f_n(\tau) := \begin{cases} (2k_n^- + 1)\pi & \text{for } \tau < -T_0 - 2, \\ u_n(\eta_n + \tau) & \text{for } \tau \in [-T_0 - 1, T_0 + 1], \\ (2k_n^+ + 1)\pi & \text{for } \tau > T_0 + 2 \end{cases} \quad (28)$$

and interpolate linearly on $[-T_0 - 2, -T_0 - 1]$ and $[T_0 + 1, T_0 + 2]$. In analogy to P_n , we introduce, replacing u_n by f_n in (21), $\tilde{P}_n : (0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \tilde{P}_n(T) := & \frac{c^2}{2} \int_{-\frac{1}{2}-T+s}^{\frac{1}{2}} \int_{-T+s}^{T+s} [f_n'(\tau)]^2 d\tau ds - \frac{c_0^2}{2} \int_{-T}^T [f_n(\tau + \frac{1}{2}) - f_n(\tau - \frac{1}{2})]^2 d\tau \\ & + K \int_{-T}^T [1 + \cos(f_n(\tau))] d\tau. \end{aligned} \quad (29)$$

For $|\tau| > T_0 + \frac{5}{2}$, each of the integrands vanishes because f_n is on $\{\tau \in \mathbb{R} : |\tau| > T_0 + 2\}$ by definition constant and equal to an odd multiple of π . Therefore

$$\tilde{P}_n(T_0 + \frac{5}{2}) = \tilde{P}_n(T) = \tilde{P}_n(\infty) = J(f_n) \quad \text{for all } T > T_0 + \frac{5}{2}. \quad (30)$$

The goal is now to show that, up to a subsequence, $J(f_n) \rightarrow \alpha \in (0, \theta)$ for $n \rightarrow \infty$. To do so, we are going to estimate, with l from (24),

$$|\tilde{P}_n(T_n) - l| = \left| [\tilde{P}_n(T_n) - P_n(T_n; \eta_n)] - [P_n(T_n; \eta_n) - l] \right|$$

in terms of ε . We know already $|P_n(T_n; \eta_n) - l| \leq \varepsilon$ from (25), so we are left with $|\tilde{P}_n(T_n) - P_n(T_n; \eta_n)|$. Note that $P_n(T_0 + \frac{1}{2}; \eta_n) = \tilde{P}_n(T_0 + \frac{1}{2})$ because, by definition, $f_n(\tau) = u_n(\eta_n + \tau)$ for all $|\tau| \leq T_0 + 1$. Thus the triangle inequality yields

$$|P_n(T_n; \eta_n) - \tilde{P}_n(T_n)| \leq |P_n(T_n; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n)| + |\tilde{P}_n(T_0 + \frac{1}{2}) - \tilde{P}_n(T_n)|. \quad (31)$$

It follows from (25) that the first term on the right-hand side of (31) can be estimated as

$$|P_n(T_n; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n)| \leq 2\varepsilon. \quad (32)$$

As for the second term on the right-hand side of (31), observe that the domains of integration in $\tilde{P}_n(T_0 + \frac{1}{2})$ and $\tilde{P}_n(T_n)$ overlap so that, with $\mathcal{I}_n := (-T_n, -T_0 - \frac{1}{2}) \cup (T_0 + \frac{1}{2}, T_n)$,

$$\begin{aligned} \left| \tilde{P}_n(T_0 + \frac{1}{2}) - \tilde{P}_n(T_n) \right| &= \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\substack{(-T_n+s, -T_0-\frac{1}{2}+s) \\ \cup (T_0+\frac{1}{2}+s, T_n+s)}} [f'_n(\tau)]^2 d\tau ds \\ &\quad - \frac{c_0^2}{2} \int_{\mathcal{I}_n} [f_n(\tau + \frac{1}{2}) - f_n(\tau - \frac{1}{2})]^2 d\tau + K \int_{\mathcal{I}_n} [1 + \cos(f_n(\tau))] d\tau \\ &\leq \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\substack{(-T_n+s, -T_0-\frac{1}{2}+s) \\ \cup (T_0+\frac{1}{2}+s, T_n+s)}} [f'_n(\tau)]^2 d\tau ds + K \int_{\mathcal{I}_n} [1 + \cos(f_n(\tau))] d\tau \end{aligned} \quad (33)$$

because $[f_n(\tau + \frac{1}{2}) - f_n(\tau - \frac{1}{2})]^2 \geq 0$. We estimate the terms first on the intervals which lie in \mathbb{R}^+ , and start with the first summand of (33).

Using $f'(\tau) = 0$ for $\tau > T_0 + 2$, we obtain

$$\frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{T_0+\frac{1}{2}+s}^{T_n+s} [f'_n(\tau)]^2 d\tau ds = \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{T_0+\frac{1}{2}+s}^{T_0+\frac{5}{2}+s} [f'_n(\tau)]^2 d\tau ds$$

(the domain of integration is here the largest parallelogram shown in Figure 3) and, splitting the integral at $z = T_0 + 1$ and employing $f'_n = u'_n(\eta_n + \cdot)$ on $(T_0, T_0 + 1)$ (this is used on the left half of the shaded parallelogram in Figure 3) and $f'_n = 0$ on $(T_0 + 2, T_0 + 3)$ (thus no contribution comes from the rightmost triangle in Figure 3)

$$= \frac{c^2}{2} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{T_0+\frac{1}{2}+s}^{T_0+1} [u'_n(\eta_n + \tau)]^2 d\tau ds + \int_{T_0+1}^{T_0+2} [f'_n(\tau)]^2 d\tau \right]$$

and, extending the first integral to the whole shaded parallelogram of Figure 3, while evaluating the second integral, in which $|f'| \leq \delta$ a.e. due to (26),

$$\leq \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta_n+T_0+\frac{1}{2}+s}^{\eta_n+T_0+\frac{3}{2}+s} [u'_n(\tau)]^2 d\tau ds + \frac{c^2}{2} \cdot 1 \cdot \delta^2$$

and, bounding the remaining integral by $\frac{2}{c^2 - c_0^2} [P_n(T_0 + \frac{3}{2}; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n)]$ by (21) and (22),

$$\leq \frac{c^2}{c^2 - c_0^2} [P_n(T_0 + \frac{3}{2}; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n)] + \frac{1}{2} \cdot c^2 \cdot \delta^2$$

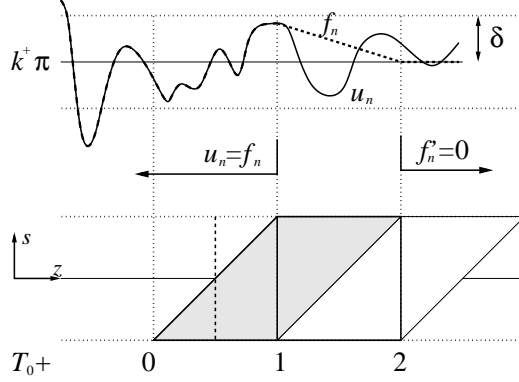


FIGURE 3. Illustration of the deduction of (34). The equality $u_n = f_n$ is to be understood modulo a translation in the argument of the two functions as described in the text.

and finally, using the monotonicity of P_n and (25),

$$\leq \frac{c^2}{c^2 - c_0^2} \cdot 2\varepsilon + \frac{1}{2} \cdot c^2 \cdot \delta^2 = c^2 \left(\frac{2\varepsilon}{c^2 - c_0^2} + \frac{\delta^2}{2} \right). \quad (34)$$

We continue estimating the right-hand side of (33) on $(T_0 + \frac{1}{2}, T_n)$ and find for the remaining third integral, from (26),

$$\left| K \int_{T_0 + \frac{1}{2}}^{T_n} [1 + \cos(f_n(\tau))] \, d\tau \right| = K \int_{T_0 + \frac{1}{2}}^{T_0 + 2} [1 + \cos(f_n(\tau))] \, d\tau \leq \frac{3}{2} K (1 - \cos(\delta)). \quad (35)$$

The very same estimates hold for the interval $(-T_n, -T_0 - \frac{1}{2})$. Thus when combining (33), (34) and (35), we obtain

$$\left| \tilde{P}_n(T_n) - \tilde{P}_n(T_0 + \frac{1}{2}) \right| \leq c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos \delta). \quad (36)$$

Therefore, using $\tilde{P}_n(T_n) = J(f_n)$ (from (30)) and inserting (32) and (36) into (31), we obtain

$$\begin{aligned} |P_n(T_n; \eta_n) - J(f_n)| &= |P_n(T_n; \eta_n) - \tilde{P}_n(T_n)| \\ &\leq 2\varepsilon + c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos \delta) =: \tilde{\varepsilon}, \end{aligned} \quad (37)$$

hence, with (25),

$$|J(f_n) - l| \leq |J(f_n) - P_n(T_n; \eta_n)| + |P_n(T_n; \eta_n) - l| \leq \tilde{\varepsilon} + \varepsilon.$$

Now choose $\varepsilon_0 > 0$ such that

$$l + 2(\tilde{\varepsilon} + \varepsilon) < \theta \quad \text{and} \quad l - 2(\tilde{\varepsilon} + \varepsilon) > 0 \quad \text{for all} \quad \varepsilon < \varepsilon_0; \quad (38)$$

this is possible because $\delta = \delta(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$ while c , c_0 and K are constants, and $0 < l < \theta$ by assumption. We will from now on assume that $\varepsilon < \varepsilon_0$.

With this choice, we find that there exists a subsequence (not relabelled) of $(f_n)_{n \in \mathbb{N}}$ for which the limit of $J(f_n)$ for $n \rightarrow \infty$ exists, and

$$\lim_{n \rightarrow \infty} J(f_n) = \alpha,$$

for some $\alpha \in \mathbb{R}$ with $|l - \alpha| \leq \tilde{\varepsilon}$ and, by choice of ε_0 , $0 < \alpha < \theta$.

Step 4. Introduce now, with k^\pm as in (26),

$$g_n(\tau) := \begin{cases} u_n(\eta_n + \tau) - 2k_n^- \pi & \text{for } \tau < -T_n + 1, \\ \pi & \text{for } \tau \in [-T_n + 2, T_n - 2], \\ u_n(\eta_n + \tau) - 2k_n^+ \pi & \text{for } \tau > T_n - 1, \end{cases} \quad (39)$$

and interpolate linearly on $[-T_n + 1, -T_n + 2]$ and $[T_n - 2, T_n - 1]$. Clearly

$$\text{dist}(\text{supp}(f'_n), \text{supp}(g'_n)) \geq (T_n - T_0 - 4) \rightarrow \infty$$

for $n \rightarrow \infty$. By definition of f_n and g_n ,

$$u_n(\eta_n + \tau) = f_n(\tau) + g_n(\tau) - \pi \quad \text{for } |\tau| \in [0, T_0 + 1] \cup [T_n - 1, \infty),$$

and for $\pm\tau \in [T_0 + 1, T_n - 1]$ we have $f_n(\tau) + g_n(\tau) - \pi = (2k^\pm + 1)\pi$, while (26) shows $|u_n - (2k^\pm + 1)\pi| \leq 2\varepsilon$.

We are now going to estimate $|J(u_n) - J(f_n) - J(g_n)|$. The last statement to be shown, $J(g_n) \rightarrow \beta \in (0, \theta)$ for $n \rightarrow \infty$, will then be an easy consequence of it. By the triangle inequality, (25) and (37),

$$\begin{aligned} |J(u_n) - J(f_n) - J(g_n)| &\leq |J(u_n) - J(g_n) - P_n(T_n - 1; \eta_n)| \\ &\quad + |P_n(T_n - 1; \eta_n) - P_n(T_n; \eta_n)| \\ &\quad + |P_n(T_n; \eta_n) - J(f_n)| \\ &\leq |J(u_n) - J(g_n) - P_n(T_n - 1; \eta_n)| + 2\varepsilon + \tilde{\varepsilon}. \end{aligned} \quad (40)$$

In a very similar manner to (33)–(36) (see also [7]), it is possible to show in analogy to (37) that

$$|J(u_n) - P(T_0; \eta_n) - J(g_n)| \leq c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos \delta) = \tilde{\varepsilon} - 2\varepsilon, \quad (41)$$

with $\tilde{\varepsilon}$ as in (37). This inequality implies together with (40)

$$|J(u_n) - J(f_n) - J(g_n)| \leq \tilde{\varepsilon} - 2\varepsilon + 2\varepsilon + \tilde{\varepsilon} = 2\tilde{\varepsilon},$$

thus for sufficiently large n

$$|J(g_n) - (\theta - \alpha)| \leq 2\tilde{\varepsilon} + \varepsilon,$$

and switching to a subsequence if necessary, we find that the limit

$$\lim J(g_n) =: \beta$$

exists, and that $0 < \beta < \theta$, by choice of ε_0 in (38). This finishes the proof. \square

5. Proof of the main result. We prove the existence of a heteroclinic wave by ruling out two of the three cases in the concentration-compactness Lemma 4.1, namely vanishing (possibility (ii)) and splitting (possibility (iii)).

Lemma 5.1 (No vanishing). *Let $c^2 > c_0^2$ and $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$ be a minimising sequence for J . Then $(u_n)_{n \in \mathbb{N}}$ does not vanish.*

Proof. The argument is by contradiction. Suppose that (20) holds. Fix $T \gg 1$, $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for all u_n with $n > N$

$$\sup_{\eta \in \mathbb{R}} \left[\int_{\eta-T}^{\eta+T} \frac{c^2}{2} [u'_n(\tau)]^2 d\tau - \int_{\eta-T}^{\eta+T-1} \frac{c_0^2}{2} [Au_n(\tau)]^2 d\tau + \int_{\eta-T}^{\eta+T} K [1 + \cos(u_n(\tau))] d\tau \right] \leq \frac{\varepsilon}{T}.$$

Observe that for supersonic waves the quantity on the left-hand side is always non-negative since the first term dominates the second one by (10), and the on-site potential $1 + \cos(\xi) \geq 0$ non-negative for every $\xi \in \mathbb{R}$. In particular,

$$\frac{c^2 - c_0^2}{2} \|u'\|_{L^2(-T, T)}^2 \leq \frac{c^2}{2} \|u'\|_{L^2(-T, T)}^2 - \frac{c_0^2}{2} \|Au\|_{L^2(-T, T-1)}^2 \leq \frac{\varepsilon}{T}$$

holds, hence $\|u'\|_{L^2(-T, T)} \leq \sqrt{\frac{4\varepsilon}{T(c^2 - c_0^2)}}$. We claim that this implies

$$|u(T) - u(-T)| \leq \sqrt{\frac{8\varepsilon}{c^2 - c_0^2}} =: \varepsilon_1. \quad (42)$$

To see this, consider the auxiliary variational problem

$$\text{Maximise } v(2T), \text{ subject to } v(0) = 0, \|v'\|_{L^2(0, 2T)} = \alpha \quad (43)$$

on $H^1(0, 2T)$. Since

$$|v(2T)| = \left| \int_0^{2T} v'(s) ds \right| \leq \|v'\|_{L^1(0, 2T)} \leq \sqrt{2T} \|v'\|_{L^2(0, 2T)},$$

where equality holds only for $v' = \text{const.}$, we conclude that the maximum must be attained by the linear functions $v(s) = \pm \frac{\alpha s}{\sqrt{2T}}$, which yields $v(2T) = \alpha \sqrt{2T}$ as maximum.

Since J is invariant under the transformation $\tau \mapsto \tau + \tau_0$ for arbitrary $\tau_0 \in \mathbb{R}$, we may assume without loss of generality that $u_n(0) = 0$ for all $n \in \mathbb{N}$. It follows

$$K \int_{-T}^T [1 + \cos(u(\tau))] d\tau \geq K (1 + \cos(\varepsilon_1)) \cdot 2T.$$

This, however, is a contradiction to

$$\sup_{\eta \in \mathbb{R}} \int_{\eta-T}^{\eta+T} K [1 + \cos(u(\tau))] d\tau < \varepsilon$$

for ε small enough. \square

Lemma 5.2 (No splitting). *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$ be a minimising sequence for J . Then, for every c with $c^2 > \frac{9}{8}c_0^2$, $(u_n)_{n \in \mathbb{N}}$ does not split in the sense of Lemma 4.1 (iii).*

Proof. The condition $c^2 > \frac{9}{8}c_0^2$ ensures that $\sqrt{\frac{c^2}{c^2 - c_0^2}} < 3$, thus $k \in \{0, 1\}$ in the statement of Lemma 3.2, so any minimiser u_0 of $J|_{\mathcal{M}_{-\pi, \pi}}$ satisfies $\|u_0\|_{L^\infty(\mathbb{R})} < 3\pi$. By the same arguments as in the proof of Lemma 3.2, the same $L^\infty(\mathbb{R})$ -bound

applies to the members of a minimising sequence (possibly after dropping finitely many of them); indeed, if $(J(u_n) - \inf J|_{\mathcal{M}_{-\pi,\pi}}) < \delta$ then, in analogy to (17),

$$8(2\kappa + 1)\sqrt{(c^2 - c_0^2)K} \leq J(u_n) \leq 8c\sqrt{K} + \delta,$$

so if $\sqrt{\frac{c^2}{c^2 - c_0^2}} < 3$, as guaranteed by assumption, then $\frac{8\sqrt{c^2 K} + \delta}{8\sqrt{(c^2 - c_0^2)K}} < 3$ for δ small enough.

Suppose now, for a contradiction, that $(u_n)_{n \in \mathbb{N}}$ splits. Let $\varepsilon > 0$ and choose, as in the proof of Lemma 4.1 (iii), f_n, g_n such that $|u_n - (f_n + g_n - \pi)| < \varepsilon$. Then the L^∞ -bound $\|u_n\|_{L^\infty(\mathbb{R})} < 3\pi$ shows $k_n^\pm \in \{-1, 0\}$, see Equation (26). For given k_n^+ , and k_n^- , we can immediately determine the values of $f_n(\infty) - f_n(-\infty)$ respectively $g_n(\infty) - g_n(-\infty)$ from (28) respectively (39); recall the boundary conditions $u_n(\infty) - u_n(-\infty) = 2\pi$. This straightforward calculation yields

$$\begin{aligned} & f_n(\infty) - f_n(-\infty) = 0, \quad g_n(\infty) - g_n(-\infty) = 2\pi \quad (\text{if } k_n^+ = k_n^- \in \{-1, 0\}), \\ \text{or } & f_n(\infty) - f_n(-\infty) = 2\pi, \quad g_n(\infty) - g_n(-\infty) = 0 \quad (\text{if } k_n^+ = 0, k_n^- = -1), \\ \text{or } & f_n(\infty) - f_n(-\infty) = -2\pi, \quad g_n(\infty) - g_n(-\infty) = 4\pi \quad (\text{if } k_n^+ = -1, k_n^- = 0). \end{aligned}$$

Thus, let \tilde{f}_n be defined by $\tilde{f}_n(\tau) := f_n(-\tau)$. Then for each $n \in \mathbb{N}$ one of the three functions f_n, g_n and \tilde{f}_n belongs to $\mathcal{M}_{-\pi,\pi}$. Let this function be denoted by \tilde{u}_n . As both

$$\lim_{n \rightarrow \infty} J(f_n) < \inf_{u \in \mathcal{M}_{-\pi,\pi}} J(u) \quad \text{and} \quad \lim_{n \rightarrow \infty} J(g_n) < \inf_{u \in \mathcal{M}_{-\pi,\pi}} J(u),$$

by Lemma 4.1 (iii), there exists a subsequence of $(\tilde{u}_n)_{n \in \mathbb{N}}$, not relabelled, such that

$$\lim_{n \rightarrow \infty} J(\tilde{u}_n) < \inf_{u \in \mathcal{M}_{-\pi,\pi}} J(u) = \lim_{n \rightarrow \infty} J(u_n),$$

a contradiction to the fact that $(u_n)_{n \in \mathbb{N}}$ is a minimising sequence in $\mathcal{M}_{-\pi,\pi}$. \square

Now we are in a position to prove the main theorem 2.2.

Proof. The Modica-Mortola bound (13) implies that J is bounded from below on X . Let $(u_n)_{n \in \mathbb{N}}$ be a minimising sequence in $\mathcal{M}_{-\pi,\pi}$. We know from Lemma 4.1 that $(u_n)_{n \in \mathbb{N}}$ has to concentrate, or vanish, or split. Lemma 5.1 rules out vanishing, and Lemma 5.2 rules out splitting for $c > \frac{9}{8}c_0^2$. Thus $(u_n)_{n \in \mathbb{N}}$ must concentrate. Hence, for fixed $\varepsilon > 0$, it is possible to choose a sequence $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $T_0 > 0$ such that

$$|J(u_n) - J_{T_0}(u_n; \eta_n)| < \varepsilon.$$

We write $w_n(\tau) := u_n(\eta_n + \tau)$. The sequence $(w_n)_{n \in \mathbb{N}}$ is bounded in X , since $\|w'_n\|_{L^2(\mathbb{R})} = \|u'_n\|_{L^2(\mathbb{R})} \leq \frac{2}{c^2 - c_0^2} J(u_n)$ and $|w(0)| < 3\pi$ by (15) and Lemma 3.2. As X is a Hilbert space, there exists a subsequence, not relabelled, which converges weakly. On the interval $[-T_0, T_0]$, the weak convergence of $(w_n)_{n \in \mathbb{N}}$ implies strong convergence in $L^2(-T_0, T_0)$ and $C^0[-T_0, T_0]$ to some limit u . Hence for all $n > N$, N sufficiently large,

$$\left| \int_{-T_0}^{T_0-1} \left[(Aw_n(\tau))^2 - (Au(\tau))^2 \right] d\tau \right| < \varepsilon$$

and

$$\left| \int_{-T_0}^{T_0} [\cos(w_n(\tau)) - \cos u(\tau)] d\tau \right| < \varepsilon.$$

As the weak convergence implies $\|u'\|_{L^2(-T_0, T_0)}^2 \leq \liminf_{n \in \mathbb{N}} \|w'_n\|_{L^2(-T_0, T_0)}^2$, we may conclude $J_{T_0}(u) \leq \liminf_{n \in \mathbb{N}} J_{T_0}(w_n; 0)$.

We now extend the domain of u inductively to \mathbb{R} . Take any monotone sequence $T_k \rightarrow \infty$ with $k \in \mathbb{N}_0$ and assume that u has already been defined as uniform limit of $(w_n)_{n \in \mathbb{N}}$ on the interval $[-T_k, T_k]$. As $(w_n)_{n \in \mathbb{N}}$ is still bounded in X we can again choose a subsequence, not relabelled, which converges uniformly in $C^0[-T_{k+1}, T_{k+1}]$ to some limit \tilde{u} , which, by construction, coincides with u on $[-T_k, T_k]$.

It follows that this function u on \mathbb{R} satisfies the boundary conditions (5) and, with a constant $C = C(c, c_0, K)$,

$$\begin{aligned} J(u) &= \lim_{T \rightarrow \infty} J_T(u; 0) \leq \lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} J_T(w_n; 0) \\ &\leq \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} J(w_n) + C\varepsilon = \lim_{n \rightarrow \infty} J(u_n) + C\varepsilon; \end{aligned}$$

in particular $u' \in L^2(\mathbb{R})$, thus $u \in M_{-\pi, \pi}$. As ε was arbitrary, the previous inequality shows

$$J(u) \leq \liminf_{n \rightarrow \infty} J(u_n).$$

This means that u is a minimiser of J on $M_{-\pi, \pi}$, and the theorem follows from the fact that (2) is the Euler-Lagrange equation of J (in the sense of Lemma 2.1). \square

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